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Orthomodular Lattice in Relativistic Non-Quantum Mechanics

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In 1975–1980, W. Cegla and A. Z. Jadczyk studied the causality structure of space-time: two points of Minkowski space-time M are causally independent iff they are different and spacelike or lightlike separated, and the measurable causally closed subsets of M form an orthomodular lattice. We show that this lattice enables us to model, by a formalism close to the one of orthodox quantum mechanics, a definite experiment in relativistic non-quantum mechanics: the counting of identical point bodies by one or several radars in some particular regions of the space-time.

The orthomodular lattice considered in this paper has been known for a long time. Our aim is to show that it may be used to model a precise experiment in relativistic non-quantum mechanics, in a way closely related to the logicoalgebraic approach to quantum mechanics.

1. PREVIOUS RESULTS

From 1975–1980, W. Cegla and A. Z. Jadczyk of the Institute of Theoretical Physics, Wroclaw, studied the causal logic of Minkowski space-time (Cegla and Jadczyk, 1977, 1979; Cegla, 1981). Their results were used in a paper by Banai (1985).

1.1. Some Definitions

Minkowski space-time M is an affine space over the vector space $\mathbf{R}^4$ equipped with the Minkowski quadratic form Q defined by

$$Q(x, y, z, t) = x^2 + y^2 + z^2 - t^2$$

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where the light velocity is $c = 1$.

Two points u, v in M are:

- *Spacelike* (resp. *timelike, lightlike*) separated if $Q(u - v) > 0$ [resp. $Q(u - v) < 0$, $Q(u - v) = 0$].
- *Causally independent* (Cegla and Jadczyk) if they are different and spacelike or lightlike separated.

A *space- or light-like hypersurface* is a (maximal) hypersurface S in M, any two points of which are spacelike or lightlike separated.

1.2. Some Results by Cegla and Jadczyk

The relation of causal independence is an orthogonality relation, denoted by $\perp$. If we define, as usual, for any subset A of M,

$$A^{\perp} = \{u \in M: \forall v \in A, u \perp v\}$$

then the set L of all subsets A of M such that $A = A^{\perp\perp}$, ordered by inclusion and equipped with the unary operation by which each A is associated $A^{\perp}$, is a complete orthomodular lattice.

If A is a Borel subset of M, then $A^{\perp}$ is a Borel set, too, and the set L_B of all Borel sets in L is a σ-complete subortholattice of L. Both L and L_B are atomic and do not satisfy the covering law.

In both cases (L and L_B), there is a one-to-one correspondence between blocks and space- or light-like hypersurfaces: atoms of a block are one-element subsets of the associated hypersurface.

Cegla and Jadczyk (1979) showed that a smooth conserved current on M, of compact support, defines a σ-additive state on L_B.

2. FINITE-VALUED STATES ON L_B

We define a universe line as any possible trajectory in M of a classical massive particle. We make the assumption that such a line C is characterized as follows:

(a) Any two different points of C are timelike separated (this means that, in any interval of time, the mean velocity of the particle is $< c$).

(b) C intersects every space- or light-like hypersurface (since the future light cone of any point of M is a space- or light-like hypersurface, this exclude the case of a particle accelerated in such a way that it cannot be reached by a light ray).

Proposition 1. There is a one-to-one correspondence between two-valued σ-additive states on L_B (or on L) and universe lines. The two-valued state s associated with such a line C is defined, for any A in L_B, by

$$s(A) = 1 \quad \text{if} \quad C \text{ intersects } A$$
$$= 0 \quad \text{otherwise}$$

Moreover, this result can be extended to n-valued states s such that $s(a) \leq 1/n$ for any atom a, and n-uples of disjoint universe lines.

The above results suggest that L_B enables us to model the counting of identical particles in certain regions of M, the regions belonging to L_B.

Now, a first question arises: regions in L_B are of very particular shapes; is there a physical reason for this?

3. WHY THESE SHAPES?

Let us imagine a counter scanning identical particles crossing over a Borel region R in M, called its range. We observe two facts:

1. It is necessary to avoid the situation described in Fig. 1, where xyz, $x'y'z'$, $xyy'z'$ are universe lines. Indeed, in this case, the counter detects particles crossing its range R following arcs xy and $y'z'$, but the observer cannot know if there is one particle going along the universe line $xyy'z'$ or two different particles following respectively xyz and $x'y'z'$, and so is not able to count the particles.

2. Let $T(R)$ be the set of all u in space-time M such that any universe line containing u intersects R.

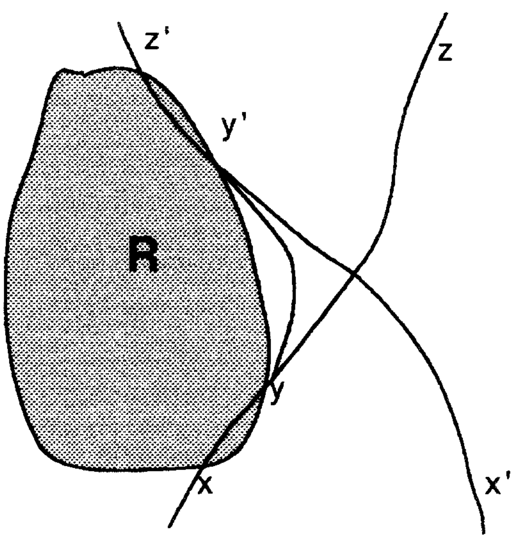

Fig. 1.

Then T is a closure operator; in particular, R is contained in $T(R)$, and the counter detects all particles crossing over $T(R)$, since the same particles cross over R and over $T(R)$.

$T(R)$ is called the full range of the counter.

Proposition 2. Condition 1 is satisfied if and only if the full range of the counter belongs to L_B.

This proves that this counter is able to count only if its full range is in L_B. However, this condition is not sufficient to be sure that the counting of particles crossing R is possible, because it can happen that the intersection with R of the trajectory in M of a particle is a nonconnected set.

For any subset A of M, let us denote by $S(A)$ the set of all u in M such that there exists a universe line containing u and meeting A in both the past and the future of u.

Proposition 3. The mapping S is a closure operation such that, for any Borel subset A of M, $S(A)$ is a Borel set, and for any A in L, $S(A) = A$. Moreover, for any subset A of M, $TS(A) = A^{\perp\perp}$.

This shows that if the range R of the counter is such that $R = S(R)$, then $T(R) = R^{\perp\perp}$, hence condition 1 is satisfied; moreover the intersection with R of any universe line is connected, hence the counter is able to work.

Remark. In the characterization of a universe line we have not supposed [see condition (a), Section 2], as usual, that a massive particle has at each time a velocity $v < c$. If the definition of a universe line is modified in this way, Propositions 2 and 3 remain true; the main consequence of this modification is that some σ-additive two-valued states are not associated with a universe line in this new sense.

4. THE COUNTER

A counter satisfying the above conditions is actually well known, but its range is very large and it detects only extended bodies: it is radar.

Let us consider a radar acting as transmitter-receiver in all directions, moving in a large interstellar space, during an interval of time. Its purpose is to detect and to count identical bodies crossing its range. These bodies and the radar are very small in relation to its range, and so they may be considered as points. The radar is supposed to be powerful enough to detect each body to which the radio waves can go and return while it works.

Proposition 4. The range of this radar is full and belongs to L_B. It is the least upper bound in L_B of two atoms $\{u\}$ and $\{v\}$ of L_B, where u, v are, respectively, the initial and final positions of the radar in the space-time.

Moreover, it is possible to use, as a counter, several radars, some of them acting as transmitters, and the others as receivers, some additional requirements being necessary in order to satisfy above condition $R = S(R)$. This allows us to obtain various shapes for the full range of the counter.

In particular, if each radar works during an interval of time, condition $R = S(R)$ is satisfied in the following circumstances.

Let us denote by $T \alpha T'$ the relation between receivers T and T', meaning that, while working, T sends to T' a radio message that T' receives while working; then the transitive relation generated by α on the set of all receivers is a universal one (that is to say, this relation holds for any two different receivers); moreover, the same is true for transmitters.

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